

MTH250 - CALCULUS

Lecture No. 30

DR. ABDUL WAHAB

Assistant Professor,
Department of Mathematics, CIIT, Pakistan.

© **VIRTUAL CAMPUS**

COMSATS Institute of Information Technology
CIIT, Virtual Campus Secretariat 402, Street 34, I-8/2 Islamabad-Pakistan.
Ph:+92 51 9101237-39 | info@vcomsats.edu.pk | vcomsats.edu.pk

These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

Contents

Contents	i
30 Surface integrals	1
30.1 Surface area & parametric representations	1
30.1.1 Parametric representation of surfaces	2
30.2 Introduction to surface integrals	5
30.2.1 Interpretation of surface integrals	5
30.3 Evaluating surface integrals	6
30.3.1 Piecewise smooth surfaces	9
30.4 Oriented surfaces	11
30.5 Flux	12

LECTURE 30

Surface integrals

30.1 Surface area & parametric representations

Definition 30.1 (Surface area). *Consider a surface σ of the form $z = f(x, y)$ defined over a region R in the xy -plane. Assume that f has continuous first partial derivatives at the interior points of R . Then, the surface area of σ is given by*

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

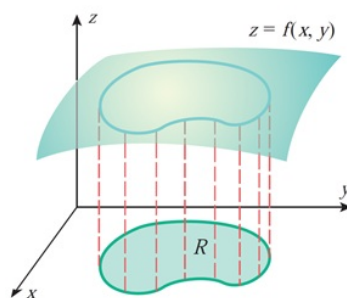


Figure 30.1: Surface area.

Example 30.1. Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy -plane whose coordinates satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 4$.

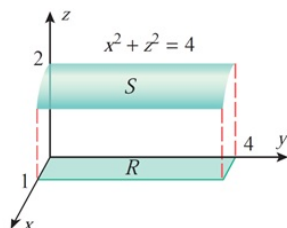


Figure 30.2: Surface $x^2 + y^2 = 4$.

Solution. As the surface is a portion of the cylinder $x^2 + z^2 = 4$ the surface area is

$$\begin{aligned} S &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA = \iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0 + 1} dA \\ &= \int_0^4 \int_0^1 \frac{2}{\sqrt{4-x^2}} dx dy = 2 \int_0^4 \left[\sin^{-1}\left(\frac{1}{2}x\right) \right]_{x=0}^1 dy \\ &= 2 \int_0^4 \frac{\pi}{6} dy = \frac{4}{3}\pi. \end{aligned}$$

□

30.1.1 Parametric representation of surfaces

A surface in $\mathbb{R}^3$ can be represented parametrically by three equations involving two parameters, say

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

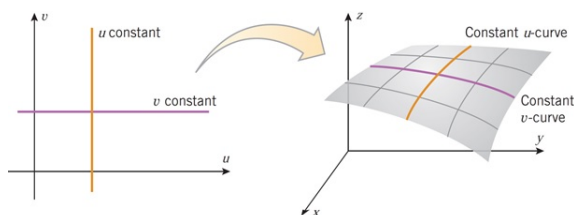


Figure 30.3: Constant u and constant v -curves

Indeed, if we fix a parameter, say u then the curve $x = x(u, v), y = y(u, v)$ depends only on v and is called a constant u -curve. Similarly, if we fix a v then the curve

$x = x(u, v), y = y(u, v)$ depends only on u and is called a constant v -curve. These curves altogether form the surface; see Figure 30.3.

Example 30.2. Consider the paraboloid $z = 4 - x^2 - y^2$. One way to parametrize this surface is to take $x = u$ and $y = v$; as the parameters, in which case the surface is represented by the parametric equation

$$x = u, \quad y = v, \quad z = 4 - x^2 - y^2.$$

We can also parametrize the surface using cylindrical coordinates. Let $x = r \cos \theta$, $y = r \sin \theta$, thus $z = 4 - r^2$. Thus the paraboloid can be represented parametrically in terms of r and θ as

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = 4 - r^2.$$

Notice that the surface $z = 4 - x^2 - y^2$ parametrized by

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2$$

can also be written as

$$\mathbf{r} = \mathbf{r}(u, v) = u\hat{i} + v\hat{j} + (4 - u^2 - v^2)\hat{k}.$$

Thus, we can conclude that the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}.$$

Definition 30.2 (Principle normal).

If a parametric surface σ is the graph of $\mathbf{r} = \mathbf{r}(u, v)$, and if

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}$$

at a point on the surface, then the principle unit normal vector to the surface at that point is denoted by $\mathbf{n}$ or $\mathbf{n}(u, v)$ and is defined as

$$\mathbf{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}.$$

Example 30.3. Find an equation of the tangent plane to the parametric surface

$$x = uv, \quad y = u, \quad z = v^2$$

at the point where $u = 2$ and $v = -1$.

Solution. We start by writing the equations in the vector form

$$\mathbf{r} = uv\hat{i} + u\hat{j} + v^2\hat{k}.$$

The partial derivatives of $\mathbf{r}$ are

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) = v\hat{i} + \hat{j}, \quad \frac{\partial \mathbf{r}}{\partial v}(u, v) = u\hat{i} + 2v\hat{k}$$

and at $u = 2$ and $v = -1$ these partial derivatives are

$$\frac{\partial \mathbf{r}}{\partial u}(2, -1) = -\hat{i} + \hat{j}, \quad \frac{\partial \mathbf{r}}{\partial v}(2, -1) = 2\hat{i} - 2\hat{k}$$

Thus, the normal to the surface at this point is

$$\frac{\partial \mathbf{r}}{\partial u}(2, -1) \times \frac{\partial \mathbf{r}}{\partial v}(2, -1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = -2\hat{i} - 2\hat{j} - 2\hat{k}.$$

Since any normal will suffice to find that tangent plane, it makes sense to multiply this vector by $-\frac{1}{2}$ and use the simpler normal $\hat{i} + \hat{j} + \hat{k}$. It follows from the given parametric equations that the point on the surface corresponding to $u = 2$ and $v = -1$ is $(-2, 2, 1)$ so the tangent plane at this point can be expressed in point-normal form as

$$(x + 2) + (y - 2) + (z - 1) = 0 \quad \text{or} \quad x + y + z = 1.$$

□

Definition 30.3 (Area of parametric surfaces).

Consider a surface σ over the region R in xy -plane with vector equation

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}.$$

Assume that $\mathbf{r}$ has continuous first partial derivatives at the interior points of R . Then, the surface area of σ is given by

$$S = \iint_R \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA.$$

Example 30.4. Let $x = u$, $y = u \cos v$, $z = u \sin v$ represent the cone that results when the line $y = x$ in the xy -plane is revolved about the x -axis. Find the surface area of the portion of the cone for which $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$.

Solution. The surface can be expressed in vector form as

$$\mathbf{r} = u\hat{i} + u \cos v \hat{j} + u \sin v \hat{k}, \quad (0 \leq u \leq 2, 0 \leq v \leq 2\pi)$$

This,

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= \hat{i} + \cos v \hat{j} + \sin v \hat{k}, & \frac{\partial \mathbf{r}}{\partial v} &= -u \sin v \hat{j} + u \cos v \hat{k}.\end{aligned}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \cos v & \sin v \\ 0 & -u \sin v & u \cos v \end{vmatrix} = u \hat{i} - u \cos v \hat{j} - u \sin v \hat{k}.$$

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{u^2 + (-u \cos v)^2 + (-u \sin v)^2} = |u| \sqrt{2} = u \sqrt{2}$$

Thus

$$S = \iint_R \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA = \int_0^{2\pi} \int_0^2 \sqrt{2} u du dv = 2\sqrt{2} \int_0^{2\pi} dv = 4\pi\sqrt{2}.$$

□

30.2 Introduction to surface integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.

We start with parametric surfaces and deal with the special case where σ is the graph of a function of two variables.

Suppose a surface σ has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}, \quad (u, v) \in D.$$

Assume that the parameter domain D is a rectangle and we divide it into sub-rectangles R_{ij} with dimensions Δu and Δv .

We evaluate f at a point P_{ij}^* in each patch, multiply by the area $\Delta\sigma_{ij}$ of the patch, and form the Riemann sum

$$\sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta\sigma_{ij}.$$

Then, we take the limit as the number of patches increases and define the surface integral of f over the surface σ as:

$$\iint_{\sigma} f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta\sigma_{ij}.$$

30.2.1 Interpretation of surface integrals

Let us see what does it mean to integrate a function along a surface.

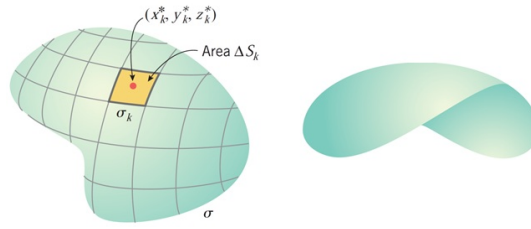


Figure 30.4: Partitioning of a surface into patches

Suppose a thin sheet (say, aluminum foil) has a shape of a surface σ . Let the density (mass per unit area) at the point (x, y, z) be $f(x, y, z)$. Then, the *total mass* of the sheet is

$$M = \iint_{\sigma} f(x, y, z) dS$$

When $f(x, y, z) = 1$, then

$$M = \iint_{\sigma} dS = \lim_{m, n \rightarrow \infty} \Delta\sigma_k = S.$$

30.3 Evaluating surface integrals

Theorem 30.4. Let σ be a smooth parametric surface whose vector equation is

$$\mathbf{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}, \quad (u, v) \in D$$

where (u, v) varies over a region R in the uv -plane. If $f(x, y, z)$ is continuous on σ , then

$$\iint_{\sigma} f(x, y, z) dS = \iint_R f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA.$$

Example 30.5. Evaluate the surface integral $\iint_{\sigma} x^2 dS$ over the sphere $x^2 + y^2 + z^2 = 1$.

Solution. As $\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \hat{i} + \sin \phi \sin \theta \hat{j} + \cos \phi \hat{k}$, where $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$, and

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sin \phi.$$

From the $\hat{i}$ -component of $\mathbf{r}$, the integrand in the surface integral can be expressed in terms of ϕ and θ as $x^2 = \sin^2 \phi \cos^2 \theta$. Thus, it follows that

$$\begin{aligned} \iint_{\sigma} x^2 dS &= \iint_R (\sin^2 \phi \cos^2 \theta) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA = \int_0^{2\pi} \int_0^{\pi} \sin^3 \phi \cos^2 \theta d\phi d\theta. \\ &= \int_0^{2\pi} \left[\int_0^{\pi} \sin^3 \phi d\phi \right] \cos^2 \theta d\theta = \int_0^{2\pi} \left[\frac{1}{3} \cos^3 \phi - \cos \phi \right]_0^{\pi} \cos^2 \theta d\theta \\ &= \frac{4}{3} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{4}{3} \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{4\pi}{3}. \end{aligned}$$

□

In the case where surface σ is of the form $z = f(x, y)$, we can take $x = u$ and $y = v$ as parameters and express the equation of the surface as $\mathbf{r} = u\hat{i} + v\hat{j} + g(u, v)\hat{k}$. In which case

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 + 1}$$

Thus

$$\iint_{\sigma} f(x, y, z) dS = \iint_R f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 + 1} dA.$$

Therefore, we have the following theorem .

Theorem 30.5.

1. Let σ be surface with equation $z = g(x, y)$ and let R be its projection over xy -plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ , then

$$\iint_{\sigma} f(x, y, z) dS = \iint_R f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 + 1} dA.$$

2. Let σ be surface with equation $y = g(x, z)$ and let R be its projection over xz -plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ , then

$$\iint_{\sigma} f(x, y, z) dS = \iint_R f(x, g(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x} \right)^2 + \left(\frac{\partial y}{\partial z} \right)^2 + 1} dA.$$

3. Let σ be surface with equation $x = g(y, z)$ and let R be its projection over yz -plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ , then

$$\iint_{\sigma} f(x, y, z) dS = \iint_R f(g(y, z), y, z) \sqrt{\left(\frac{\partial x}{\partial y} \right)^2 + \left(\frac{\partial x}{\partial z} \right)^2 + 1} dA.$$

Example 30.6. Evaluate the surface integral $\iint_{\sigma} xz ds$ where σ is the part of the plane $x + y + z = 1$ that lies in the first octant.

Solution.

The equation of the plane can be written as $z = 1 - x - y$. Consequently, with $z = g(x, y) = 1 - x - y$ and $f(x, y, z) = xz$. We have

$$\frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = -1,$$

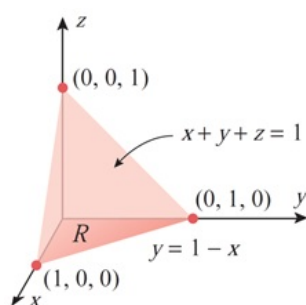


Figure 30.5: The part of the plane $x + y + z = 1$ that lies in the first octant.

so

$$\iint_{\sigma} xz dS = \iint_R x(1-x-y) \sqrt{(-1)^2 + (-1)^2 + 1} dA.$$

where R is the projection of σ on the xy -plane. Rewriting the double integral as an iterated integral yields

$$\begin{aligned} \iint_{\sigma} xy dS &= \sqrt{3} \int_0^1 \int_0^{1-x} (x - x^2 - xy) dy dx \\ &= \sqrt{3} \int_0^1 \left[xy - x^2 y - \frac{xy^2}{2} \right]_{y=0}^{1-x} dx \\ &= \sqrt{3} \int_0^1 \left[\frac{x}{2} - x^2 - \frac{xy^2}{2} \right] dx \\ &= \sqrt{3} \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \right]_0^1 = \frac{\sqrt{3}}{24}. \end{aligned}$$

□

Example 30.7. Evaluate the surface integral $\iint_{\sigma} y^2 z^2 dS$, where σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$.

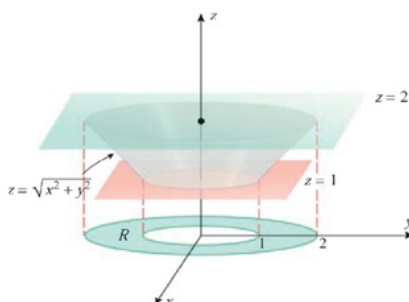


Figure 30.6: The part of the cone $z = \sqrt{x^2 + y^2}$ that lies in the between the planes $z = 1$ and $z = 2$.

Solution. We have

$$z = g(x, y) = \sqrt{x^2 + y^2} \quad \text{and} \quad f(x, y, z) = y^2 z^2.$$

Thus

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

So,

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2},$$

yields

$$\iint_{\sigma} y^2 z^2 dS = \iint_R y^2 \left(\sqrt{x^2 + y^2}\right)^2 \sqrt{2} dA = \sqrt{2} \iint_R y^2 (x^2 + y^2) dA$$

where R is the annulus enclosed between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Using the polar coordinates to evaluate this double integral over the annulus R yields

$$\begin{aligned} \iint_{\sigma} y^2 z^2 dS &= \sqrt{2} \int_0^{2\pi} \int_1^2 (r \sin \theta)^2 (r^2) r dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_1^2 r^5 \sin^2 \theta dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{r^6}{6} \sin^2 \theta \right]_{r=1}^2 d\theta \\ &= \frac{21}{\sqrt{2}} \int_0^{2\pi} \sin^2 \theta d\theta \\ &= \frac{21}{\sqrt{2}} \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}}. \end{aligned}$$

□

30.3.1 Piecewise smooth surfaces

If σ is a piecewise-smooth surface *i.e.* a finite union of smooth surfaces $\sigma_1, \sigma_2, \dots, \sigma_n$ that intersect only along their boundaries, then the surface integral of f over σ is defined by:

$$\iint_{\sigma} f(x, y, z) dS = \iint_{\sigma_1} f(x, y, z) dS + \iint_{\sigma_2} f(x, y, z) dS + \dots + \iint_{\sigma_n} f(x, y, z) dS$$

Example 30.8. Evaluate $\iint_S z ds$ where S is the surface whose: side S_1 is given by the cylinder $x^2 + y^2 = 1$, Bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$. Top S_3 is the part of the plane $z = 1 + x$ that lies above S_2 .

Solution. For S_1 , we use θ and z as parameters and write its parametric equations as

$$0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1 + x = 1 + \cos \theta.$$

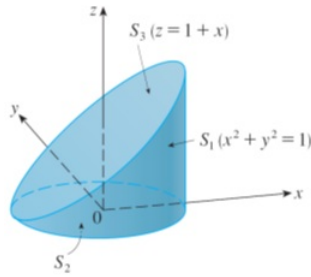


Figure 30.7: Surface whose: side S_1 is given by the cylinder $x^2 + y^2 = 1$, Bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$. Top S_3 is the part of the plane $z = 1 + x$ that lies above S_2 .

Therefore,

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta \hat{i} + \sin\theta \hat{j}$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Thus the surface integral over S_1 is

$$\begin{aligned} \iint_{S_1} z dS &= \iint_D z |\mathbf{r}_\theta \times \mathbf{r}_z| dA = \int_0^{2\pi} \int_0^{1+\cos\theta} z dz d\theta \\ &= \int_0^{2\pi} (1+\cos\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} \left[1 + 2\cos\theta + \frac{1}{2}(1+\cos\theta) \right] d\theta \\ &= \frac{1}{2} \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{3\pi}{2}. \end{aligned}$$

Since S_2 lies in the plane $z = 0$, we have,

$$\iint_{S_2} z dS = \iint_{S_2} 0 dS = 0.$$

The surface S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So,

taking $g(x, y) = 1 + x$ in converting to polar coordinates, we have

$$\begin{aligned}
 \iint_{S_3} z dS &= \iint_D (1+x) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\
 &= \int_0^{2\pi} \int_0^1 (1+r \cos \theta) \sqrt{1+1+0} r dr d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \int_0^1 (r + r^2 \cos \theta) dr d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3} \cos \theta \right) d\theta \\
 &= \sqrt{2} \left[\frac{\theta}{2} + \frac{\sin \theta}{3} \right]_0^{2\pi} = \sqrt{2} \pi
 \end{aligned}$$

Finally, we have

$$\iint_S z dS = \iint_{S_1} z dS + \iint_{S_2} z dS + \iint_{S_3} z dS = \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \left(\frac{3}{2} + 0 + \sqrt{2} \right) \pi.$$

□

30.4 Oriented surfaces

Most surfaces that we encounter in applications have two sides *e.g.* a sphere has an inside and an outside.

There exist mathematical surfaces with only one side *e.g.* Möbius strip, which has only one side in the sense that a bug can traverse the entire surface without crossing an edge.

A sphere is two-sided in the sense that a bug walking on the sphere can traverse the inside surface or the outside surface but cannot traverse both without somehow passing through the sphere.

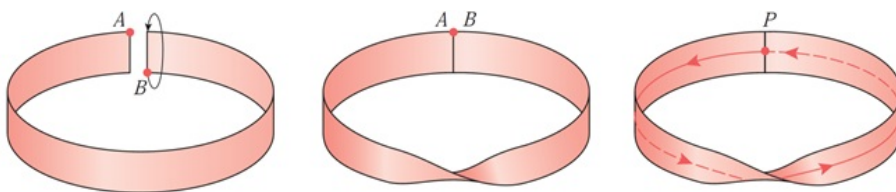


Figure 30.8: One sided surfaces: Möbius strip:

Definition 30.6 (Orientable surfaces).

A two sided surface is said to be orientable and a one sided surface is said to be nonorientable.

If σ is an orientable surface that has a unit normal vector $\mathbf{n}$ at each point. Then, the vectors $\mathbf{n}$ and $-\mathbf{n}$ point to opposite sides of the surface and distinguish between the two sides.

If σ is a smooth orientable surface, then it is always possible to choose the direction of $\mathbf{n}$ at each point so that $\mathbf{n} = \mathbf{n}(x, y, z)$ varies continuously over the surface.

These unit vectors are said to form the orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

where

$$\mathbf{r}_s = \frac{\partial x}{\partial s} \hat{i} + \frac{\partial y}{\partial s} \hat{j} + \frac{\partial z}{\partial s} \hat{k}$$

We call this the positive orientation of the parametric surface and we say that $\mathbf{n}$ points in the positive direction from the surface.

30.5 Flux

Definition 30.7 (Flux).

The volume of fluid that passes through the surface in one unit of time. The flux of a force $\mathbf{F}$ across a surface σ with orientation $\mathbf{n}$ is the quantity Φ defined by

$$\Phi = \iint_{\sigma} \mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z) dS.$$

Theorem 30.8. Let σ be a smooth parametric surface represented by the vector equation $\mathbf{r} = \mathbf{r}(u, v)$ in which (u, v) varies over a region R in the uv -plane. If the component functions of the vector field $\mathbf{F}$ are continuous on σ , and if $\mathbf{n}$ determines the positive orientation of σ , then

$$\Phi = \iint_{\sigma} \mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z) dS = \iint_R \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$$

where it is understood that the integrand on the right side of the equation is expressed in terms of u and v .

Example 30.9. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\hat{k}$ across the outward oriented sphere $x^2 + y^2 + z^2 = a^2$.

Solution. The sphere with outward positive orientation can be expressed by the vector valued function

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \hat{i} + a \sin \phi \sin \theta \hat{j} + a \cos \phi \hat{k}$$

where

$$0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

Therefore,

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = a^2 \sin^2 \phi \cos \theta \hat{i} + a^2 \sin^2 \phi \sin \theta \hat{j} + a^2 \sin \phi \cos \phi \hat{k}.$$

Moreover, for the points on the sphere, we have $\mathbf{F} = z\hat{k} = a \cos \phi \hat{k}$. Hence,

$$\mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) = a^3 \sin \phi \cos 2\phi$$

Thus, it follows with the parameters u and v replaced by ϕ and θ that

$$\begin{aligned} \Phi &= \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) \\ &= \int_0^{2\pi} \int_0^{\pi} a^3 \sin \phi \cos^2 \phi d\phi d\theta = a^3 \int_0^{2\pi} \left[-\frac{\cos^3 \phi}{3} \right]_0^{\pi} d\theta \\ &= \frac{2a^3}{3} \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}. \end{aligned}$$

□